ECED 3300
Tutorial 2

Problem 1

Determine gradients of the following functions

a) $f(x, y, z) = xyz$*;*

b) $f(x, y, z) = x^2 + y^2 + z^2$.

Solution

a) By definiition, $\nabla f = \mathbf{a}_x \partial_x f + \mathbf{a}_y \partial_y f + \mathbf{a}_z \partial_z f = yz\mathbf{a}_x + xz\mathbf{a}_y + xy\mathbf{a}_z$.

b) In the Cartesian coordinates, $\nabla f = \mathbf{a}_x \partial_x f + \mathbf{a}_y \partial_y f + \mathbf{a}_z \partial_z f = 2(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) = 2\mathbf{r}$. In the spherical coordinates, $f = x^2 + y^2 + z^2 = r^2$. Thus, $\nabla f = \mathbf{a}_r \partial_r r^2 = 2r\mathbf{a}_r = 2\mathbf{r}$.

Problem 2

The temperature in a room is distributed according to $T(x, y, z) = x^2 + y^2 - z$*. A mosquito, located at* $(1, 1, 2)$ *in the room, seeks to fly in the direction of the fastest temperature increase. Which direction will it fly in?*

Solution

Recall that the direction of the fastest growth/decay of a scalar field at a given point is determined by the field gradient direction at the point. By definition,

$$\nabla T = \mathbf{a}_x \partial_x T + \mathbf{a}_y \partial_y T + \mathbf{a}_z \partial_z T = 2x\mathbf{a}_x + 2y\mathbf{a}_y - \mathbf{a}_z.$$

Thus at $(1, 1, 2)$,

$$\nabla T(1, 1, 2) = 2\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z; \qquad |\nabla T(1, 1, 2)| = 3.$$

Hence the direction sought by the mosquito is specified by the unit vector,

$$\mathbf{a}_s = \frac{\nabla T(1, 1, 2)}{|\nabla T(1, 1, 2)|} = \frac{1}{3}(2\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z).$$

Problem 3

Show that a unit normal to the surface specified by $z = f(x, y)$ is given by the expression,

$$\mathbf{a}_n = \frac{-\mathbf{a}_x \partial_x f - \mathbf{a}_y \partial_y f + \mathbf{a}_z}{\sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}}$$

Solution

The equation for a surface can be rewritten in the form,

$$F(x, y, z) = z - f(x, y) = 0.$$

The gradient of a scalar field F is known to be normal to any surface of constant F, in particular, to $F(x, y, z) = 0$. Hence a normal to the surface coincides with a normalized gradient of F. The latter follows from the definition to be

$$\nabla F = \mathbf{a}_x \partial_x F + \mathbf{a}_y \partial_y F + \mathbf{a}_z \partial_z F = -\mathbf{a}_x \partial_x f - \mathbf{a}_y \partial_y f + \mathbf{a}_z.$$

Further,

$$|\nabla F| = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}.$$

Thus,

$$\mathbf{a}_n = \frac{\nabla F}{|\nabla F|} = \frac{-\mathbf{a}_x \partial_x f - \mathbf{a}_y \partial_y f + \mathbf{a}_z}{\sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}}$$

Q.E.D.

Problem 4

Given $\mathbf{E} = \mathbf{a}_x yz + \mathbf{a}_y xz + \mathbf{a}_z xy$ and a hemisphere of radius R in the upper half-space $z > 0$, rimmed by a circle in the xy-plane, find the flux through the hemisphere.

Solution

The hemisphere plus the circle form a closed surface to which a Gauss' theorem is applicable,

$$\oint_S d\mathbf{S} \cdot \mathbf{E} = \int dv \nabla \cdot \mathbf{E}.$$

By definition,

$$\nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z = \partial_x(yz) + \partial_y(xz) + \partial_z(xy) = 0.$$

Hence,

$$\oint_S d\mathbf{S} \cdot \mathbf{E} = 0.$$

Problem 5

Determine the flux $\Phi = \int d\mathbf{S} \cdot \mathbf{F}$ *in the following situations:*

a) $\mathbf{F} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$ *and the surface is three squares of side* b *in the* xy*,* xz *and* yz *planes intersecting at the origin.*

b) $\mathbf{F} = (\mathbf{a}_x x + \mathbf{a}_y y)\ln(x^2 + y^2)$ *and the surface is a cylinder of radius* R *and height* h *with its bottom in the* xy*-plane and its axis coinciding with the* z*-axis.*

c) $\mathbf{F} = (\mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z)e^{-(x^2+y^2+z^2)}$ *and the surface is a sphere of radius* R*, centered at the origin.*

Solution

a) It is sufficient to work out the field on the surfaces $z = 0$, $y = 0$ and $x = 0$:

- at $z = 0$, $\mathbf{a}_n = \mathbf{a}_z$, $\mathbf{F} \cdot \mathbf{a}_n = F_z(z = 0) = 0$;
- By symmetry, at $x = 0$, $\mathbf{a}_n = \mathbf{a}_x$, $\mathbf{F} \cdot \mathbf{a}_n = F_x(x = 0) = 0$, and at $y = 0$, $\mathbf{a}_n = \mathbf{a}_y$, $\mathbf{F} \cdot \mathbf{a}_n = F_y(y = 0) = 0$.

Thus,

$$\Phi = 0.$$

b) At the top and bottom $\mathbf{a}_n = \pm\mathbf{a}_z$. It follows that $\mathbf{F} \cdot \mathbf{a}_n = 0$. On the wall, $\rho = R$, $\mathbf{a}_n = \mathbf{a}_\rho$, and $dS = Rd\phi dz$. Hence,

$$\Phi = \int_0^h dz \int_0^{2\pi} d\phi\, [R^2 \ln R(\mathbf{a}_\rho \cdot \mathbf{a}_x)\cos\phi + R^2 \ln R(\mathbf{a}_\rho \cdot \mathbf{a}_y)\sin\phi] = hR^2 \ln R \int_0^{2\pi} d\phi(\cos^2\phi + \sin^2\phi) = 2\pi h R^2 \ln R.$$

c) On the sphere $r = R$, $\mathbf{a}_n = \mathbf{a}_r$ and $dS = R^2 \sin\theta d\theta d\phi$. Also, it is convenient to shift to the spherical coordinates such that

$$\mathbf{F} = R(\mathbf{a}_x \sin\theta\cos\phi + \mathbf{a}_y \sin\theta\sin\phi + \mathbf{a}_z \cos\theta)e^{-R^2}.$$

It follows that

$$\Phi = R^3 e^{-R^2} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi(\mathbf{a}_x \sin\theta\cos\phi + \mathbf{a}_y \sin\theta\sin\phi + \mathbf{a}_z \cos\theta) \cdot \mathbf{a}_r$$

Using the fact that

$$\mathbf{a}_r \cdot \mathbf{a}_x = \sin\theta\cos\phi, \qquad \mathbf{a}_r \cdot \mathbf{a}_y = \sin\theta\sin\phi, \qquad \mathbf{a}_r \cdot \mathbf{a}_z = \cos\theta\,.$$

we obtain after elementary trigonometry that

$$\Phi = R^3 e^{-R^2} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, (\sin^2\theta + \cos^2\theta) = 4\pi R^3 e^{-R^2}.$$